

Generalization of Some Algebras in the Bosonic String Theory

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Abstract We assume that the total target phase space is non-commutative. This leads to the generalization of the oscillator-algebra of the string, and the corresponding Virasoro algebra. The effects of this non-commutativity on some string states will be studied.

Keywords Bosonic string · Non-commutativity

1 Introduction

The non-commutative geometry [1] has been considered for some time in connection with various physics subjects. Recent motivation to study the non-commutative geometry mainly comes from the string theory. String theories have been pointing towards a non-commuting scenario already in the 80's [2]. Various subjects in the non-commutativity in string theory can be found in the Refs. [3–18]. We are interesting to the non-commutative world-sheet of the Bosonic string, e.g. see Refs. [17, 18]. Thus, we study the string propagation in the non-commutative phase space. That is, we consider the following commutation relations

$$\begin{aligned} [X^\mu(\sigma, \tau), \Pi^\nu(\sigma', \tau)] &= i\eta^{\mu\nu}\delta(\sigma - \sigma'), \\ [X^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] &= i\theta^{\mu\nu}(\sigma - \sigma'), \\ [\Pi^\mu(\sigma, \tau), \Pi^\nu(\sigma', \tau)] &= i\gamma^{\mu\nu}(\sigma - \sigma'), \end{aligned} \quad (1)$$

where $\Pi^\mu(\sigma, \tau) = \frac{1}{2\pi\alpha'}\partial_\tau X^\mu(\sigma, \tau)$ is the canonical momentum, conjugate to $X^\mu(\sigma, \tau)$. The variable $\theta^{\mu\nu}$ indicates the non-commutativity of the space part, and $\gamma^{\mu\nu}$ shows the non-commutativity of the momentum part of the phase space.

This paper is organized as follows. In Sect. 2, the generalized oscillator algebra will be obtained. In Sect. 3, the associated Virasoro algebra will be studied. Section 4 is devoted to the conclusions.

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2 Oscillator Algebra

The Fourier expansions of the variables $X^\mu(\sigma, \tau)$, $\Pi^\mu(\sigma, \tau)$, $\theta^{\mu\nu}(\sigma - \sigma')$ and $\gamma^{\mu\nu}(\sigma - \sigma')$ are as in the following

$$\theta^{\mu\nu}(\sigma - \sigma') = \sum_{n=-\infty}^{\infty} \theta_n^{\mu\nu} e^{in(\sigma-\sigma')}, \tag{2}$$

$$\gamma^{\mu\nu}(\sigma - \sigma') = \sum_{n=-\infty}^{\infty} \gamma_n^{\mu\nu} e^{in(\sigma-\sigma')}, \tag{3}$$

$$X^\mu(\sigma, \tau) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \cos n\sigma e^{-in\tau}, \tag{4}$$

where, for simplicity we consider the open string solution.

Introducing the mode expansions (2)–(4) in (1) gives the following oscillator-algebra

$$\begin{aligned} [p^\mu, p^\nu] &= i\pi^2 \gamma_0^{\mu\nu}, \\ [x^\mu, p^\nu] &= i\eta^{\mu\nu} - 2i\pi^2 \alpha' \tau \gamma_0^{\mu\nu}, \\ [x^\mu, x^\nu] &= i\theta_0^{\mu\nu} - 4i\pi^2 \alpha'^2 \tau^2 \gamma_0^{\mu\nu}, \end{aligned} \tag{5}$$

for the zero-modes, and

$$[\alpha_m^\mu, \alpha_n^\nu] = \left(m\eta^{\mu\nu} + 2i\alpha'\pi^2 \gamma_n^{\mu\nu} + \frac{n^2}{2\alpha'} \theta_n^{\mu\nu} \right) \delta_{n+m,0}, \tag{6}$$

for the oscillating-modes.

We observe that the non-commutativity of the phase space modifies the algebra. In other words, even if its space part is commutative, *i.e.* $\theta^{\mu\nu} = 0$, the parameter $\gamma_0^{\mu\nu}$ tells us that zero-modes of the space part is non-commutative. However, the oscillating algebra is affected by both non-commutativity parameters $\theta^{\mu\nu}$ and $\gamma^{\mu\nu}$. Vanishing $\theta^{\mu\nu}$ and $\gamma^{\mu\nu}$ implies the usual algebra for the string modes, as expected.

2.1 Conditions on the Non-Commutativity Parameters

Now take the Hermitean conjugate of the both sides of the second and third equations of (1), This leads to the equations

$$[\theta^{\mu\nu}(\sigma - \sigma')]^\dagger = -\theta^{\nu\mu}(\sigma' - \sigma), \tag{7}$$

$$[\gamma^{\mu\nu}(\sigma - \sigma')]^\dagger = -\gamma^{\nu\mu}(\sigma' - \sigma). \tag{8}$$

In terms of the oscillating modes, we obtain

$$\begin{aligned} (\gamma_n^{\mu\nu})^\dagger &= -\gamma_n^{\nu\mu}, \\ (\theta_n^{\mu\nu})^\dagger &= -\theta_n^{\nu\mu}. \end{aligned} \tag{9}$$

That is, effect of the Hermitean conjugation from changing the mode index “ n ” of α_n^μ has been modified to the exchange of the space-time indices μ and ν .

3 The Corresponding Virasoro Algebra

We only assumed the quantization (1). Therefore, the string action does not change. This implies that the Virasoro operators remain as previous, *i.e.*,

$$L_m^{(\alpha)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n}^{\mu} \alpha_{n\mu}. \tag{10}$$

Due to the modification of the oscillator-algebra (6), the corresponding Virasoro algebra also is modified, *i.e.*,

$$[L_m^{(\alpha)}, L_n^{(\alpha)}] = (m - n)L_{m+n}^{(\alpha)} + \frac{d}{12}m(m^2 - 1)\delta_{m+n,0} + \mathfrak{F}_{mn}, \tag{11}$$

where \mathfrak{F}_{mn} is the consequence of the non-commutativity

$$\mathfrak{F}_{mn} = \frac{1}{2} \sum_{\ell=-\infty}^{\infty} \lambda_{mn,\mu\nu} \alpha_{\ell}^{\nu} \alpha_{m+n-\ell}^{\mu}, \tag{12}$$

$$\lambda_{mn}^{\mu\nu} = 2i\alpha'\pi^2(\gamma_{n-m}^{\nu\mu} + \gamma_{n-m}^{\mu\nu}) + \frac{(n - m)^2}{2\alpha'} \left(\theta_{n-m}^{\nu\mu} + \theta_{n-m}^{\mu\nu} \right). \tag{13}$$

Therefore, the second and third terms of the right-hand-side of (11) are originated from the anomaly.

3.1 The Ghosts Contribution

Introducing the conformal ghosts $b(\sigma, \tau)$ and $c(\sigma, \tau)$, the Virasoro operator takes the form

$$L_m = L_m^{(\alpha)} + L_m^{(g)}, \tag{14}$$

where

$$L_m^{(g)} = - \sum_{n=-\infty}^{\infty} (m - n)b_{m+n}c_{-n}. \tag{15}$$

According to the anti-commutation relation $\{c_m, b_n\} = \delta_{m+n,0}$ the Virasoro algebra of L_m becomes

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{d - 26}{12}m(m^2 - 1)\delta_{m+n,0} + \mathfrak{F}_{mn}. \tag{16}$$

Therefore, for the choice $d = 26$ the usual anomaly is removed. However, the \mathfrak{F}_{mn} which is anomaly due to the non-commutativity always remains.

4 Conclusions

Without modification of the string action, we assumed a total non-commutativity target phase space. This non-commutativity is induced to the oscillator algebra. Thus, we have a modified algebra. The non-commutativity of the momentum part implies that the total zero-modes of the phase space become non-commutative. However, these non-commutativity parameters are restricted by some conditions.

The Virasoro operators save their forms, as in the commutative case. The modification of oscillator-algebra induces an extra anomaly term in the Virasoro algebra.

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